H: Laplace's Equation in Spherical Coordinates

Consider

$$r^2\nabla^2 u = \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2(\phi)}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\text{for } r < 1, 0 \le \phi \le \pi, 0 \le \theta < 2\pi \tag{1}$$

$$u(1,\theta,\phi) = g(\theta,\phi) \quad \text{for } 0 \le \phi \le \pi, 0 \le \theta < 2\pi$$

You can view this as a steady-state heat conduction problem with specified heat distribution maintained on the outer boundary of the unit sphere. Since the equation seems a little intimidating, let us only consider some special cases.

Case 1: $g(\theta,\phi) = \gamma =$ constant
Then, since g does not depend on θ and ϕ, neither does u, so (1) becomes $\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0 \quad \Rightarrow \quad \frac{du}{dr} = \frac{a}{r^2} \quad \Rightarrow \quad u = -\frac{a}{r} + b = b = \gamma,$
where we have invoked the boundedness condition at $r = 0$ by setting $a = 0$.
Remark: Constants and constant$/r$ are the only solutions ("potentials") that depend only on the radial distance from the origin; $u = 1/r$ is called the **Newton potential** in physics.

Case 2: $g = g(\phi)$ only
Then (1) becomes

$$\begin{cases} \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) = 0 & r < 1, 0 \le \phi \le \pi \\ u(1,\phi) = g(\phi) & 0 \le \phi \le \pi \end{cases}$$

Let $u(r,\phi) = R(r)\Phi(\phi)$. Then

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Phi\sin(\phi)}\frac{d}{d\phi}\left(\sin(\phi)\frac{d\Phi}{d\phi}\right) = \lambda\ ,$$

which gives

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R = r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda R = 0 \tag{2}$$

again giving us a Cauchy-Euler type equation to solve on $0 \le r < 1$. Also,

$$\frac{d}{d\phi}\left(\sin(\phi)\frac{d\Phi}{d\phi}\right) + \lambda \sin(\phi)\Phi = 0 \quad \text{for } 0 \le \phi \le \pi \tag{3}$$

For (2), $R = r^\alpha$ gives characteristic equation $\alpha^2 + \alpha - \lambda = 0$ with solutions $2\alpha = -1 \pm \sqrt{1+4\lambda}$. For (3) we put it in a more standard form by letting $\Phi(\phi) = P(x)$, where $x = \cos(\phi)$. Hence, $\frac{d}{d\phi} = -\sin(\phi)\frac{d}{dx}$, so $\sin(\phi)\frac{d}{dx}\left(\sin^2(\phi)\frac{dP}{dx}\right) + \lambda\sin(\phi)P = 0$, or

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \lambda P = 0 \quad \text{for } -1 < x < 1 \; . \tag{4}$$

This is **Legendre's equation**. It is also a *singular* Sturm-Liouville equation. It can be shown through a study of series solutions that the only bounded solutions are when $\lambda = n(n+1)$, with $n = 0, 1, 2, \ldots$, and in fact the solutions to

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + n(n+1)P = 0 \tag{5}$$

are polynomials $P_n(x)$ called the **Legendre polynomials**. A scaling has been adopted that has become convention so that

$$P_0(x) \equiv 1 \; , \quad P_1(x) = x \quad . \tag{6}$$

0.1 Some Properties of Legendre Polynomials

1. **Pure recurrence relation**: The $P_n's$ are related by

   $$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x) \quad n \ge 2$$

   So, given (6), we have, for example,

   $$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},\; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x,\; P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}, \tag{7}$$
   $$P_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x \; ,$$
   $$P_6(x) = \frac{1}{16}\left(231x^6 - 315x^4 + 104x^2 - 5\right) ,$$
   $$\textit{etc.}$$

2. **Rodrigue's formula**: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n\right]$.
This can be used to obtain various properties, including the important orthogonality relation, that is

3. **Orthogonality relation**:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \begin{cases} 0 & n \neq m \\ \frac{2}{2m+1} & n = m \end{cases}$$

In spherical coordinates this gives us

$$\int_{0}^{\pi} P_n(\cos\phi) P_m(\cos\phi) \sin\phi \, d\phi = \begin{cases} 0 & n \neq m \\ \frac{2}{2m+1} & n = m \end{cases}$$

4. Note that $P_n(x)$ is an odd function of x if $n =$ odd, and an even function if n is even: $P_n(-x) = (-1)^n P_n(x)$.

0.2 Solution to Laplace's Equation in the Ball

Now, if $\lambda = n(n+1)$, from the characteristic equation for the R equation, $\alpha^2 + \alpha - \lambda = 0$, roots are $\alpha = n, -(n+1)$, so $R(r) = ar^n + br^{-(n+1)}$. For boundedness of R at $r = 0$, set $b = 0$. Thus, we have the modes $u_n(r, \phi) = r^n P_n(\cos\phi)$. Therefore,

$$u(r, \phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi) \ .$$

For the boundary condition, using (6),(7),

$$g(\phi) = \sum_{n=0}^{\infty} A_n P_n(\cos\phi) = \left(A_0 + A_1 x + \frac{A_2}{2}(3x^2 - 1) + \frac{A_3}{2}(5x^3 - 3x) + \ldots \right) \ ,$$

where $x = \cos\phi$. Hence, integrating both sides and using the orthogonality relation, we have

$$\int_0^{\pi} g(\phi) P_m(\cos\phi) \sin\phi \, d\phi = \sum_{n=0}^{\infty} A_n \int_0^{\pi} P_n(\cos\phi) P_m(\cos\phi) \sin\phi \, d\phi$$
$$= \sum_{n=0}^{\infty} A_n \int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2A_m}{2m+1}$$

which implies

$$A_m = \frac{2m+1}{2} \int_0^{\pi} g(\phi) P_m(\cos\phi) \sin\phi \, d\phi \quad \text{for } m \geq 0 \, .$$

Example: Consider

$$\begin{cases} \nabla^2 u = 0 & r < 1, \ 0 \leq \phi \leq \pi \\ u(1,\phi) = \cos(3\phi) & \end{cases}$$

Thus, in the unit ball, u does not depend on θ (so on latitude lines $u =$ constant, the constant only depending on the value of ϕ). Note that

$$\begin{aligned} \cos(3\phi) &= \cos((2+1)\phi) = \cos(2\phi)\cos(\phi) - \sin(2\phi)\sin(\phi) \\ &= (2\cos^2(\phi) - 1)\cos(\phi) - 2\sin^2(\phi)\cos(\phi) \\ &= 2\cos^3(\phi) - \cos(\phi) - 2(\cos(\phi) - \cos^3(\phi)) \\ &= 4\cos^3(\phi) - 3\cos(\phi) = 4x^3 - 3x \quad . \end{aligned}$$

But $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, so $4x^3 - 3x = \frac{8}{5}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) - \frac{3}{5}x$, so that $\cos(3\phi) = \frac{8}{5}P_3(\cos\phi) - \frac{3}{5}P_1(\cos\phi) = g(\phi)$. Therefore,

$$\begin{aligned} u(r,\phi) &= \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi) \\ &= A_0 + A_1 r P_1(\cos\phi) + A_2 r^2 P_2(\cos\phi) + A_3 r^3 P_3(\cos\phi) \\ &= -\frac{3}{5} r P_1(\cos\phi) + \frac{8}{5} r^3 P_3(\cos\phi) \quad . \end{aligned}$$

Exercise: Write the solution $u(r,\phi)$ to

$$\begin{cases} \nabla^2 u = 0 & r < 1, \ 0 \leq \phi \leq \pi \\ u(1,\phi) = \cos(4\phi) & \end{cases}$$

in an expansion in Legrendre polynomials.